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DYNAMIC COMPLIANCE MATRIX OF RIGID STRIP FOOTING BONDED TO A VISCOELASTIC CROSS ANISOTROPIC HALFSPACE

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Summary—The problem of determining the response of a rigid strip footing bonded to the surface of a viscoelastic cross-anisotropic halfspace is considered. The footing is subjected to vertical, shear and moment forces harmonically varying with time and uniformly distributed across the longitudinal axis, so that plane strain conditions prevail. The solution is based on a transformation that uncouples the wave equations in closed-form and formulates the mixed boundary condition in terms of the Green's functions for the halfspace. Characteristic results, presented in the form of dynamic compliances as functions of frequency, demonstrate the importance of the degree of cross-anisotropy and of the internal soil damping on the response.

NOTATION

A_r	$\omega B\sqrt{(\rho/E_r)}$ dimensionless frequency parameter
B	one-half of the foundation width
$A(\lambda), B(\lambda)$	integration functions of the transform parameter λ (equation 17)
$\mathbf{D}$	normalized dynamic compliance matrix (equation 1)
$\mathbf{d}$	vector of the three rigid body displacements of a foundation (equation 1)
E_V, E_H	vertical and horizontal Young's moduli of the cross-anisotropic halfspace
$\mathbf{F}$	vector of the three resultant forces, referred to the centroidal axis Oy of a foundation (equation 1)
G_{VH}	shear modulus of soil on a vertical plane
$H(x,z), N(x,z)$	wave potentials defined by equation (9)
$\mathbf{L}$	transformation matrix (equation 25)
n	E_H/E_V
$\bar{p}(\lambda)$	Fourier transform of $p(x)$
P, Q, R	soil moduli defined by equations (3) and (5)
u, W	displacements in the x, z direction
$\mathbf{U}(\omega)$	global flexibility matrix of dimensions 2(2m + 1) by 2(2m + 1) (equation 24)
$\Delta_{HH}, \Delta_{MM}, \Delta_{HM}, \Delta_V$	horizontal, rocking, coupled horizontal-rocking and vertical compliances
$\epsilon_{xx}, \epsilon_{zz}, \gamma_{xz}$	the three components of strain in plane strain deformation
ν_{VH}	Poisson's ratio for transverse strain in the horizontal direction due to a vertical stress
ν_{HH}	Poisson's ratio for transverse strain in the horizontal direction due to a horizontal stress
ξ	hysteretic damping ratio of the soil
ρ	soil density
$\sigma_{xx}, \sigma_{zz}, \tau_{xz}$	normal and shear components of the stress tensor
ω	circular frequency of excitation
$\delta_H, \delta_V, \theta$	horizontal, vertical displacement and rotation of rigid footing

Superscripts

–	Fourier transformed parameter
*	viscoelastic material constant

INTRODUCTION

The analysis of foundations continuously supported by soil and subjected to vibratory loads constitutes an important branch of modern geotechnical engineering. Considerable research, especially in recent years, has focused in developing generalized dynamic force-displacement relationships for rigid footings of circular, strip or rectangular shape bonded to idealized soil media. Such relationships are needed in the design of machine foundations the seismic analysis of soil-structure interaction and the prediction of wave-induced oscillations of off-shore platforms and caissons.

For the particular case of long structures, such as dams, caissons, long rectangular buildings, etc. it is appropriate to idealize the foundation as an infinite long strip. If the dynamic loading is uniform along the longitudinal direction, plane-strain conditions prevail throughout, and the steady state harmonic response of the foundation can be completely described through the compliance matrix $\mathbf{D}$ which relates the amplitudes of the horizontal (δ_H), vertical (δ_v) and rotational (θ) displacements with the amplitudes of the corresponding resultant forces (F_H, F_V, M) acting on the rigid footing:

$$\begin{Bmatrix} \delta_H \\ \theta \cdot B \\ \delta_v \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} \Delta_{HH} & \Delta_{HM} & 0 \\ \Delta_{MH} & \Delta_{MM} & 0 \\ 0 & 0 & \Delta_v \end{bmatrix} \begin{Bmatrix} F_H \\ \dfrac{M}{B} \\ F_v \end{Bmatrix} \tag{1a}$$

or in compact form

$$\mathbf{d} = \frac{1}{E}\mathbf{DF} \tag{1b}$$

in which E is a characteristic modulus of the soil and B is half the width of the footing. The dimensionless dynamic compliance coefficients, Δ_{HH}, Δ_{HM}, Δ_{MH} and Δ_v, are complex quantities with real and imaginary parts that are functions of the dimensionless frequency parameter

$$A = \frac{\omega B}{\sqrt{(E/\rho)}} \tag{1c}$$

in which ω is the circular frequency of vibration. (The "coupling" compliances Δ_{HM} and Δ_{MH} are equal as can be shown by use of the dynamic reciprocity theorems [1]). Determination of each compliance function, $\Delta(A)$, for a specific soil profile, calls for the solution of a mixed boundary value elastodynamic problem. Several analytical, numerical and analytical-numerical solutions have been published to date [2–7] *all of which idealize the soil as an elastic isotropic continuum.*

However, there exists abundant experimental evidence [8–16] suggesting that most natural soils and rocks posses a definite anisotropic character. This is because their fabric is intimately related to the mechanical processes occuring during their formation, which involves anisotropic stress systems. Thus, e.g. natural clay deposits formed by sedimentation and, subsequently, one-dimensional consolidation over long periods of time acquire a fabric that is characterized by particles or particle clusters oriented in a horizontal arrangement. This preferred orientation makes the clay a cross-anisotropic material with a vertical axis of symmetry. Similarly, fabric anisotropy in sands arises from the influence of gravity forces and particle shape on the deposition process, while in rocks the anisotropy may result from the anisotropy of forming minerals and/or micro- or macro-fabric features [15].

Since the number of elastic parameters required to describe the behaviour of a cross-anisotropic medium increases from 2 to 5 and "the basic equations are far more complicated than for isotropic materials..." [17], few attempts have been made to incorporate material cross-anisotropy in mixed boundary value elastodynamic problems. A particular difficulty with such materials stems from the fact that the differential equations of motion do not, in general, uncouple into classical dilatational and shear wave equations, as in the case of isotropic media. Thus no closed-form solutions can be obtained for stresses or displacements in an unbounded medium.

To overcome this difficulty, Carrier [18] in 1946 constrained the elastic anisotropic parameters to satisfy the relation

$$(P - G_{VH})(R - G_{VH}) = (Q + G_{VH})^2 \tag{2}$$

where P, Q, R, G_{VH} are defined in equations (3) and (5). The sixth order partial differential operator associated with the motion of such a constrained cross-anisotropic medium can then be factorized into three second-order operators, whereby analytical solutions can be obtained using, e.g. integral transform techniques [19].

Relationship (2) has been subsequently adopted by Cameron and Eason [20] and Payton [21] who evaluated the displacement field due to a concentrated source suddenly applied at a point in an infinite cross-anisotropic elastic solid. More recently, Kirkner [22] studied the dynamic behaviour of rigid circular plates bonded to the surface of a cross-anisotropic elastic or viscoelastic half-space whose material constants satisfied the constraint relationship (2). He considered vertical, horizontal and rocking vibrations and formulated the problem so that one stress vanished over the entire plane surface ("relaxed" boundary) while an oscillating displacement was prescribed in the loaded region. Each case led to a mixed boundary value problem represented by dual integral equations which were reduced to a single Fredholm integral equation and solved numerically. Note, finally, that equation (2) was also employed by Valliappan *et al.* [23] in order to obtain dashpot constants of an "energy absorbing" boundary that was incorporated into a plane-strain dynamic finite-element formulation aimed at studying the response of strip foundations on cross-anisotropic soils.

Clearly, establishment of equation (2) was motivated solely by the ensuing mathematical convenience, with very little consideration of its physical reality. Hence, it is rather fortunate that significant experimental evidence has come to support its validity for soils. Recently, the author has found that equation (2) is satisfied with sufficient accuracy by the elastic properties of a variety of soils, including the overconsolidated London clay [8, 14], normally consolidated kaolinitic and illitic clays [9, 12, 25] some sensitive clays [16], sands [15] and clay shales [15]. (Details of this important corroboration can be found in Refs. [24, 32].) Consequently, idealizing the soil as a constrained cross-anisotropic medium is undoubtedly justified.

In this paper an analytical-numerical method is presented to obtain the dynamic compliance matrix of a rigid strip footing bonded to the surface of a cross-anisotropic viscoelastic halfspace having a vertical axis of symmetry (Fig. 1). A formulation in terms of Green's functions for the halfspace is used instead of the more conventional dual integral equation approach, since the latter can only treat the relaxed ("smooth") boundary (as was done, e.g. in [3, 22]). Note that Luco and Westman [4] formulated the corresponding problem for the isotropic halfspace also in terms of the Green's functions, which were then used to obtain pairs of coupled Cauchy-type integral equations that were reduced to coupled Fredholm integral equations and solved numerically for a Poisson's ratio equal to 1/2 (incompressible medium). To avoid the mathematical difficulties of the above approach (which would certainly become even greater with a cross-anisotropic material) a somewhat different procedure is followed herein. Dynamic flexibility influence coefficients, defined for uniformly spaced nodal

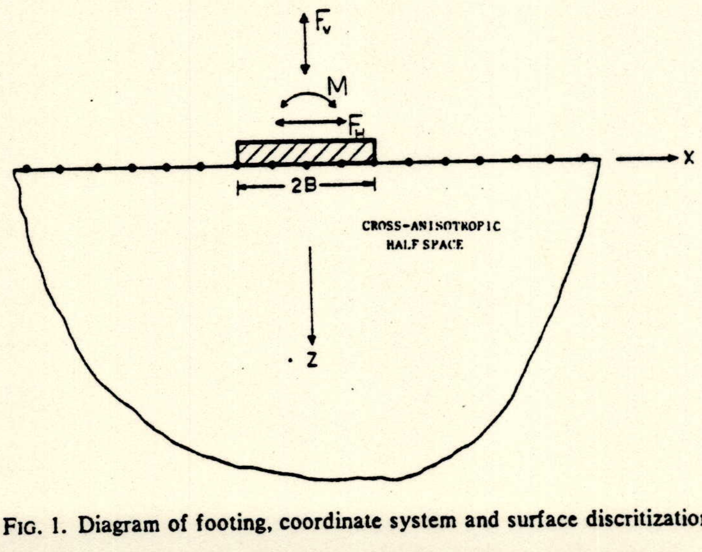

FIG. 1. Diagram of footing, coordinate system and surface discritization.

points at the surface, are obtained from solutions of two boundary value problems associated with harmonically time-varying normal or shear stresses uniformly distributed around a nodal point. Numerical evaluation of these coefficients is accomplished through a fast Fourier transform (FFT) algorithm, as was done in Ref. [6] for isotropic material and the dynamic compliance matrix of a rigid footing is subsequently evaluated by imposing the conditions of rigid body motion to the nodal points at the soil-foundation interface. Characteristic results of the method are presented in the form of plots of the four dynamic compliances versus a dimensionless frequency factor and clearly demonstrate the significance of cross-anisotropy on the response of foundations.

THE STRESS-STRAIN RELATIONS

Let the axis $0z$, which is normal the the plane surface $z = 0$ of the half space $z \geq 0$, be the axis of n-fold symmetry for the material. The horizontal planes parallel to $0xy$ are then plans of isotropy and if $0y$ is the logitudinal axis of the infinitely long footing (as in Fig. 1), the elastic stress–strain relations appropriate to plane strain deformation in the (x, z)-plane are (see, e.g. [26])

$$\sigma_{xx} = P\epsilon_{xx} + Q\epsilon_{zz} \tag{3a}$$

$$\sigma_{zz} = Q\,\epsilon_{xx} + R\,\epsilon_{zz} \tag{3b}$$

$$\tau_{xz} = G_{VH}\,\gamma_{xz} \tag{3c}$$

where the components of strain are defined in terms of the displacement components (u, w) in the directions $0x$ and $0z$ by

$$\epsilon_{xx} = u_{,x}; \quad \epsilon_{zz} = w_{,z}; \quad \gamma_{xz} = u_{,z} + w_{,x}. \tag{4}$$

The parameters P, Q and R are related to the Poisson's ratios ν_{VH}, ν^2_{HH} and Young's moduli E_V, E_H by the expressions

$$P = \frac{E_H}{a}(1 - n\nu^2_{VH}) \tag{5a}$$

$$Q = \frac{E_H}{a}\,\nu_{VH}\,(1 + \nu_{HH}) \tag{5b}$$

$$R = \frac{E_V}{a}(1 - \nu^2_{HH}) \tag{5c}$$

$$a = (1 + \nu_{HH})(1 - \nu_{HH} - 2n\nu^2_{VH});\; n = \frac{E_H}{E_V} \tag{5d}$$

while $G_{VH} = G_{HV}$ is the shear modulus in the vertical plane xz. Notice that equation (2) can be used to obtain G_{VH} in terms of the four independent material constants, ν_{VH}, ν_{HH}, E_V and E_H. The latter, however, cannot take arbitrary values as they are restricted by thermodynamic considerations [14, 24, 26] to satisfy, in general, the following inequalities:

$$-1 < \nu_{HH} < 1 - 2n\,\nu^2_{VH} \tag{6a}$$

$$\nu_{VH} < \frac{1}{\sqrt{n}}. \tag{6b}$$

Furthermore, for an incompressible material, an additional limitation is that [14]

$$n \leq 4. \tag{6c}$$

It is worth noting that although $n = 1$ is a necessary condition for this medium to respond isotropically, this alone is not sufficient; an additional requirement is that $\nu_{VH} = \nu_{HH}$ or that $G_{VH} = E_V/2(1 + \nu_{VH})$.

The constitutive equations for a simple *viscoelastic* anisotropic material can be obtained from equations (3) if the material constants, P, Q, R and G_{VH}, are replaced by complex moduli of the type

$$S^* = S\left(1 + i\,\omega\,\frac{S'}{S}\right) \tag{7a}$$

for a Voigt solid, or of the type

$$S^* = S(1 + i2\xi_S) \tag{7b}$$

for a constant hysteretic solid. S' are viscocity constants and ξ_S are damping constants independent of frequency, corresponding to modulus $S = P, Q, R$ or G_{VH}. Since the internal dissipation of energy in soils is, within the range of engineering interest, essentially independent of the frequency of excitation [27], only the second viscoelastic model (constant hysteretic solid) is considered in the sequel. The simplify the problem it is assumed that

$$\xi_P = \xi_Q = \xi_R = \xi_{G_{VH}} = \xi. \tag{7c}$$

Equation (7c) leads to Poisson's ratios, ν_{VH} and ν_{HH}, and ratio of Young's moduli, n, that are real numbers, independent of the hysteretic damping, ξ.

GOVERNING EQUATIONS AND SOLUTION

For conditions of plane strain, and a harmonic time variation $e^{i\omega t}$ of u and w, the Navier-type equations governing the motion in a cross-anisotropic viscoelastic medium are (see, e.g. [28])

$$P^* u_{,xx} + G^*_{VH} u_{,zz} + (Q^* + G^*_{VH}) w_{,xz} = -\rho\omega^2 u \tag{8a}$$

$$G^*_{VH} w_{,xx} + R^* w_{,zz} + (Q^* + G^*_{VH}) u_{,xz} = -\rho\omega^2 w. \tag{8b}$$

To uncouple equations (8), we define a pseudo-dilatational and a pseudo-distortional wave potential, $N(x, z) e^{i\omega t}$ and $H(x, z) e^{i\omega t}$ related to the displacements u and w as follows [22]:

$$u = N_{,x} + b H_{,z} \tag{9a}$$

$$w = b N_{,z} - H_{,x} \tag{9b}$$

where

$$b = \frac{R - G_{VH}}{Q + G_{VH}}. \tag{10}$$

By combining equations (2) and (10), it can easily be shown that

$$\frac{P}{R} = \frac{Q + G_{VH}}{Rb} + \frac{G_{VH}}{R} \tag{11}$$

Substituting equations (9) in (8), while accounting for (10) and (11), leads after some straight-forward operations to the following equations:

$$\frac{\partial}{\partial x}\left(\frac{P}{R} N_{,xx} + N_{,zz}\right) + \frac{\partial}{\partial z}(H_{,xx} + H_{,zz}) b \frac{G_{VH}}{R} = -\frac{\rho\omega^2}{R^*}\frac{\partial}{\partial x}(N) - \frac{\rho\omega^2}{R^*}\frac{\partial}{\partial z}(bH) \tag{12a}$$

$$\frac{\partial}{\partial z}\left(\frac{P}{R} N_{,xx} + N_{,zz}\right) b - \frac{\partial}{\partial x}(H_{,xx} + H_{,zz}) \frac{G_{VH}}{R} = -\frac{\rho\omega^2}{R^*}\frac{\partial}{\partial z}(bN) + \frac{\rho\omega^2}{R^*}\frac{\partial}{\partial x}(H). \tag{12b}$$

Equations (12) are directly reduced to two *uncoupled equations in terms of* N and H:

$$\frac{P}{R} N_{,xx} + N_{,zz} = -h^2 N \tag{13a}$$

$$H_{,xx} + H_{,zz} = -k^2 H \tag{13b}$$

where

$$h^2 = \rho\omega^2/R^*, \quad k^2 = \rho\omega^2/G^*_{VH}. \tag{14}$$

It is easier to solve equations (13) by transforming them into ordinary second-order differential equations in only the variable z, after introducing the complex Fourier transform of the dependent variables:

$$\bar{N}(\lambda, z) = \int_{-\infty}^{\infty} N(x, z) \exp(i\lambda x)\, dx \tag{15a}$$

$$\bar{H}(\lambda, x) = \int_{-\infty}^{\infty} H(x, z) \exp(i\lambda x)\, dx. \tag{15b}$$

Applying equations (15) to (13) transforms the partial differential equations to

$$\frac{d^2\bar{N}}{dz^2} - \left(\frac{P}{R}\lambda^2 - h^2\right)\bar{N} = 0 \tag{16a}$$

$$\frac{d^2\bar{H}}{dz^2} - (\lambda^2 - k^2)\bar{H} = 0 \tag{16b}$$

which present a convenient formulation of the governing differential equations. Considering the fact that as the depth of the medium, z, becomes infinite the transforms of all response quantities vanish identically, the general solution of equations (16) is

$$\bar{N} = A(\lambda)\exp(-a_1 z) \tag{17a}$$

$$\bar{H} = B(\lambda)\exp(-a_2 z) \tag{17b}$$

where $A(\lambda)$, $B(\lambda)$ are arbitrary functions of the transform parameter λ to be determined from the boundary conditions, and

$$a_1^2 = \frac{P}{R}\lambda^2 - h^2 \tag{18a}$$

$$a_2^2 = \lambda^2 - k^2 \tag{18b}$$

with $\mathrm{Re}[a_1] > 0$ and $\mathrm{Re}[a_2] > 0$.

SOLUTION OF TWO BOUNDARY VALUE PROBLEMS—FLEXIBILITY INFLUENCE COEFFICIENTS

Dynamic flexibility influence coefficients are defined as the frequency dependent displacements at any point on the surface of the soil due to a harmoninally varying with time unit vertical or unit horizontal line load applied at the origin, $x = 0$. Only the formulation for the first boundary value problem (applied vertical load) is outlined herein, since the formulation of the second problem is completely analogous.

The transformed boundary conditions at the surface ($z = 0$), when a normal traction $p(x)$ is centered at the origin of coordinate axes, are

$$\bar{\sigma}_{zz}(\lambda, 0) = \bar{p}(\lambda) = \int_{-\infty}^{\infty} p(x)\exp(i\lambda x)\,dx \tag{19a}$$

$$\bar{\tau}_{xz}(\lambda, 0) = 0 \tag{19b}$$

where $\bar{\sigma}_{zz}$ and $\bar{\tau}_{xz}$ denote the complex Fourier transforms of normal and shear stresses acting on the surface. Using equations (3), (4), (7), (9) and (17) and taking into account the well known properties of the Fourier transforms [19], it is found that

$$\bar{\sigma}_{zz}(\lambda, z) = (a_1^2 bR^* - \lambda^2 Q^*)A(\lambda)\,e^{-a_1 z} - i\lambda a_2(R^* - bQ^*)\,B(\lambda)\,e^{-a_2 z} \tag{20a}$$

$$\bar{\tau}_{xz}(\lambda, z) = i a_1 \lambda(1 + b)G^*_{VH}\,A(\lambda)\,e^{-a_1 z} + (ba_2^2 + \lambda^2)G_{VH}\,B(\lambda)\,e^{-a_2 z}. \tag{20b}$$

Substitution of (20) into (19) yields the two integration functions $A(\lambda)$ and $B(\lambda)$ that define the wave potentials $\bar{N}(\lambda, z)$ and $\bar{H}(\lambda, z)$; the transformed displacements $\bar{U}(\lambda, z)$, $\bar{W}(\lambda, z)$ are then derived from (9). By applying the inverse Fourier transform to $\bar{U}(\lambda, 0)$, $\bar{W}(\lambda, 0)$ the surface displacements are obtained:

$$u(x, 0) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \lambda\bar{p}(\lambda)\frac{ba_2^2 + \lambda^2 - a_1 a_2 b(1+b)}{F(\lambda)}e^{-i\lambda x}\,d\lambda \tag{21a}$$

$$w(x, 0) = \frac{a_1}{2\pi}\int_{-\infty}^{\infty} \bar{p}(\lambda)\frac{b(ba_2^2 + \lambda^2) - \lambda^2(1+b)}{F(\lambda)}e^{-i\lambda x}\,d\lambda \tag{21b}$$

in which

$$F(\lambda) = \lambda^2 a_1 a_2(1+b)\cdot(R^* - bQ^*) - (ba_2^2 + \lambda^2)\cdot(ba_1^2 R^* - \lambda^2 Q^*) \tag{22}$$

is the Rayleigh function for the cross-anisotropic halfspace. It is interesting to notice that equation (22) reduces to the Rayleigh function of the isotropic halfspace (see, e.g. [1 or 28]), in the special case of $n = 1$ and $\nu_{VH} = \nu_{HH} = \nu$, i.e. when the medium behaves isotropically. Indeed, in such a case

$$P = R;\ \frac{Q}{R} = \frac{\nu}{1-\nu};\ \frac{G_{VH}}{R} = \frac{1-2\nu}{2(1-\nu)};\ b = 1;\quad a_1^2 = \lambda^2 - h^2$$

and it is a simple matter of albebra to show that equation (22) takes the form

$$F(\lambda) = G^*[4\lambda^2 a_1 a_2 - (2\lambda^2 - k^2)^2] \tag{23}$$

i.e. simplifies to the isotropic Rayleigh function [1, 28].

Note, furthermore, that equation (23) agrees with the Rayleigh function determined by Kirkner [22] for a similarly constrained cross-anisotropic half-space [his equation (2.24)].

The numerical scheme proposed in Ref. [6] for isotropic layered media has been adopted herein to evaluate $u(x, 0)$ and $w(x, 0)$ from equations (21). The soil surface $z = 0$ is represented by a set of M equidistant points at which displacements (flexibility coefficients) are to be determined for a unit normal or shear pulse centered at point O. Taking M to be a power of 2 (e.g. 256, 512, or 1024), a discrete Fast Fourier Transform (FFT) algorithm [29] can be used in place of the integrals (19a) and (21). Details on the accuracy of the technique and the requirements for obtaining efficient solutions (minimum value of M, appropriate spacing between nodal points) can be found in Ref. [6]. Note that, recently, Dasqupta and Chopra [7] employed a somewhat similar procedure to evaluate the dynamic flexibility coefficients associated with a viscoelastic *isotropic* halfspace; however, they performed a direct numerical evaluation of the corresponding integrals using Simpson's rule rather than utilizing a FFT algorithm.

EVALUATION OF COMPLIANCE MATRIX FOR RIGID FOOTING

For the problem at hand the foundation-soil interface is represented by $2m+1$ (m = integer) of the M equidistant surface grid points, as shown in Fig. 1. Having determined, as outlined above, the dynamic flexibility influence coefficients due to a normal or shear unit stress pulse acting at the origin, one can readily assemble a global flexibility matrix $\mathbf{U}(\omega)$ dimensions $2(2m+1)$ by $2(2m+1)$ relating the two displacement components with the two force components acting at each node under the foundation. Notice that the computations need not be repeated for each point since, as the applied traction moves from the origin O to another point, the displacements at all points just shift by the same amount. Calling $\mathbf{U}$ and $\mathbf{P}$ the $2(2m+1)$ by 1 vectors of nodal displacements and nodal forces, one can write:

$$\mathbf{u} = \mathbf{U}(\omega)\mathbf{P}. \tag{24}$$

Imposing now the condition of rigid body motion for the foundation, the displacements of the mesh points, $\mathbf{u}$, are related to those of the corresponding centroid, $\mathbf{d}$ by the transformation

$$\mathbf{u} = \mathbf{L}\mathbf{d} \tag{25}$$

with

$$\mathbf{L}^T = [\mathbf{l}_0 \ldots \mathbf{l}_i \ldots \mathbf{l}_{2m+1}] \tag{25a}$$

and

$$\mathbf{l}_i = \begin{Bmatrix} 1 & 0 \\ 0 & x_i \\ 0 & 1 \end{Bmatrix} \tag{25b}$$

where x_i is the distance of the i nodal point from the origin.

The resultant forces on the foundation (F_H, M, F_V) can be obtained from the forces applied at the nodal points, $\mathbf{P}$, by the relationship

$$\mathbf{F} = \mathbf{L}^T P \tag{26}$$

Elimination of $\mathbf{u}$ and $\mathbf{P}$ from equations (24)–(26) and use of equation (1) leads to the desired dynamic compliance matrix

$$\mathbf{D} = E_V\, \mathbf{L}^{-1}\, \mathbf{U}(\omega)\, (\mathbf{L}^T)^{-1} \tag{27}$$

if the Young's modulus in the vertical direction E_V, is chosen as the "characteristic" modulus in equations (1), as was done in the parametric studies reported next.

RESULTS

Figs. 2–9 present the dynamic compliances Δ_{HH}, Δ_{MM}, Δ_{HM} and Δ_V as functions of the dimensionless freequncy factor, $A_V = \omega B\sqrt{(\rho/E_V)}$, for a medium which exhibits zero volumetric strain upon loading. Such an "*incompressible*" material is of particular interest to geotechnical engineers concerned with estimating foundation settlements caused by structural loads during or immediately after the construction period. Since clays are fully saturated with water, this so-called "immediate" settlement takes place before "consolidation" due to expulsion of water from the pore space can occur. As any water-solid mixture is essentially incompressible relative to a (porous) grain skeleton, the immediate displacements occur with practically no volume change.

Using equations (3)–(5), it is seen that in order for the volumetric strain, $\epsilon_v = \epsilon_{XX} + \epsilon_{YY} + \epsilon_{ZZ}$, to be zero regardless of imposed stresses, the following conditions should be met (see, also, Ref. [14, 24, 30])

$$\nu_{VH} = \frac{1}{2} \quad \text{and} \quad \nu_{HH} = 1 - \frac{1}{2}n. \tag{28}$$

Thus, a single parameter, the ratio $n = E_H/E_V$, can fully describe the *degree of anisotropy of the material*. Typically, n may vary between 0·6 – 2·0 [8–16, 24], although more extreme values have also been reported in a few cases [15].

Figures. 2–5 portray the effect of n on the four compliance functions associated with a rigid strip bonded to a halfspace characterized by a hysteretic damping ratio $\xi = 0{\cdot}05$. Both real and imaginary parts are plotted, for A_V values up to 4·0. The following conclusions can be drawn:

(a) As the ratio of horizontal to vertical modulus, n, increases, the halfspace becomes stiffer and the near static (i.e. for $A_V \to 0$) compliances decrease. However, this is not always true at higher frequencies, especially with modes of vibration that produce primarily compressional waves (vertical, rocking). At the limit $n = 4$ the halfspace becomes infinitely rigid and all compliance functions vanish, a result already

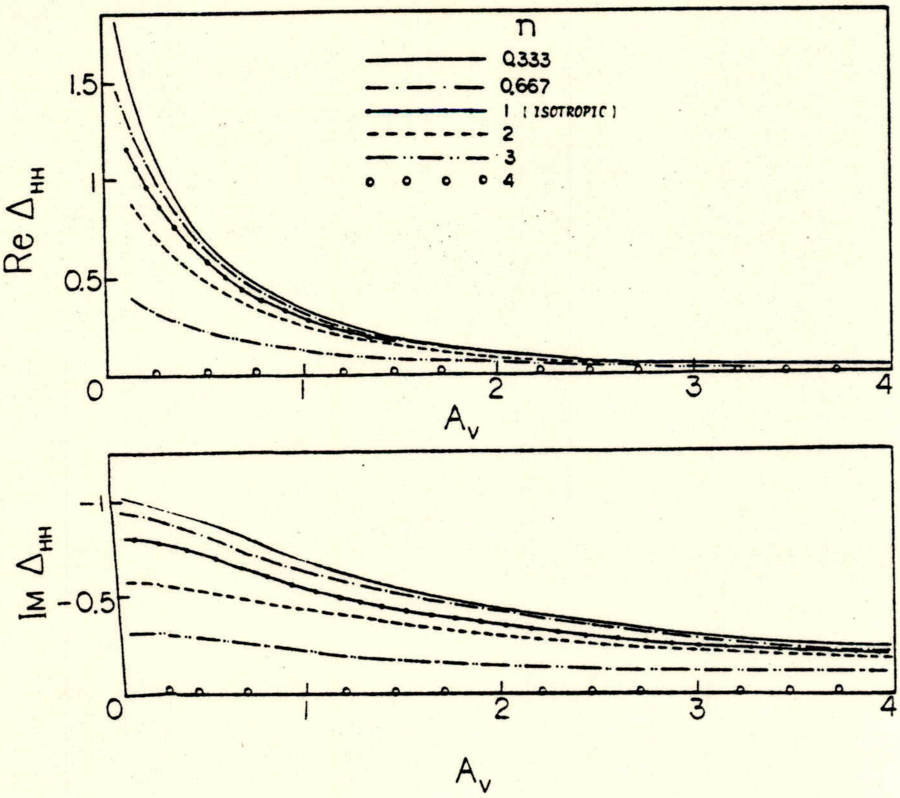

FIG. 2. Horizontal compliance for incompressible soil ($\xi = 0{\cdot}05$).

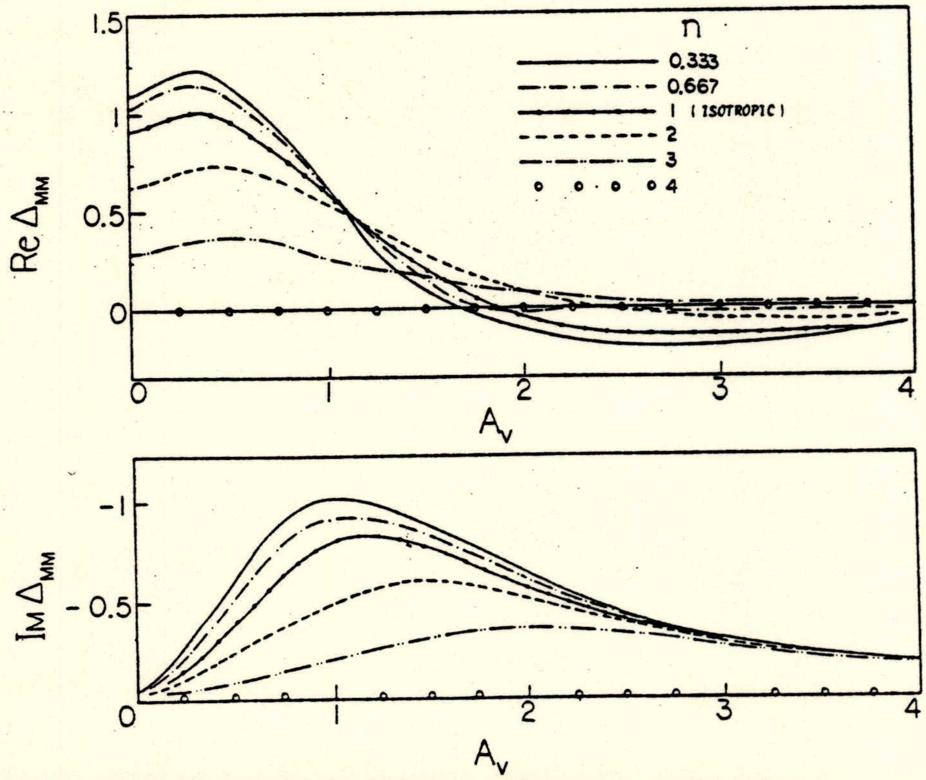

FIG. 3. Rocking compliance for incompressible soil ($\xi = 0{\cdot}05$).

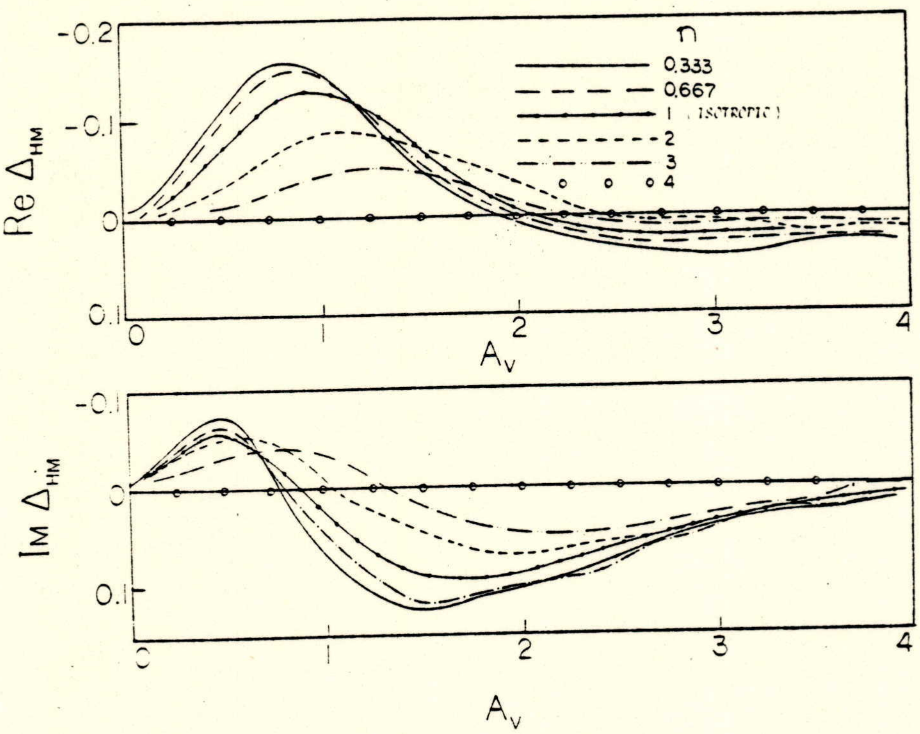

FIG. 4. Coupling compliance for incompressible soil ($\xi = 0{\cdot}05$).

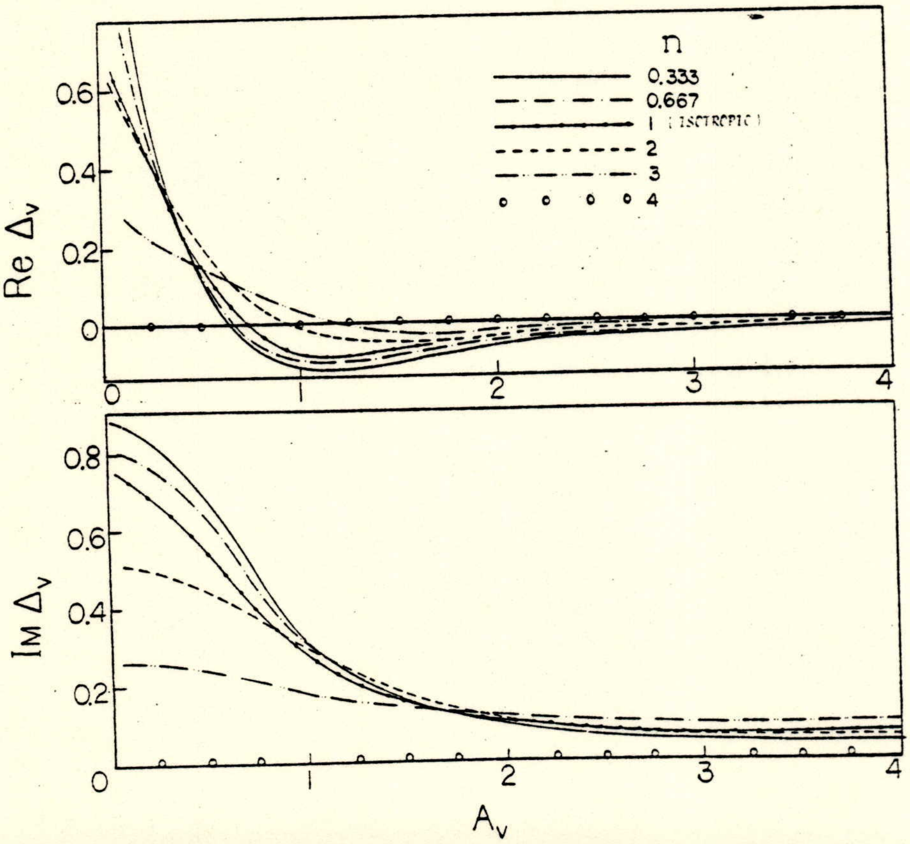

FIG. 5. Vertical compliance for incompressible soil ($\xi = 0{\cdot}05$).

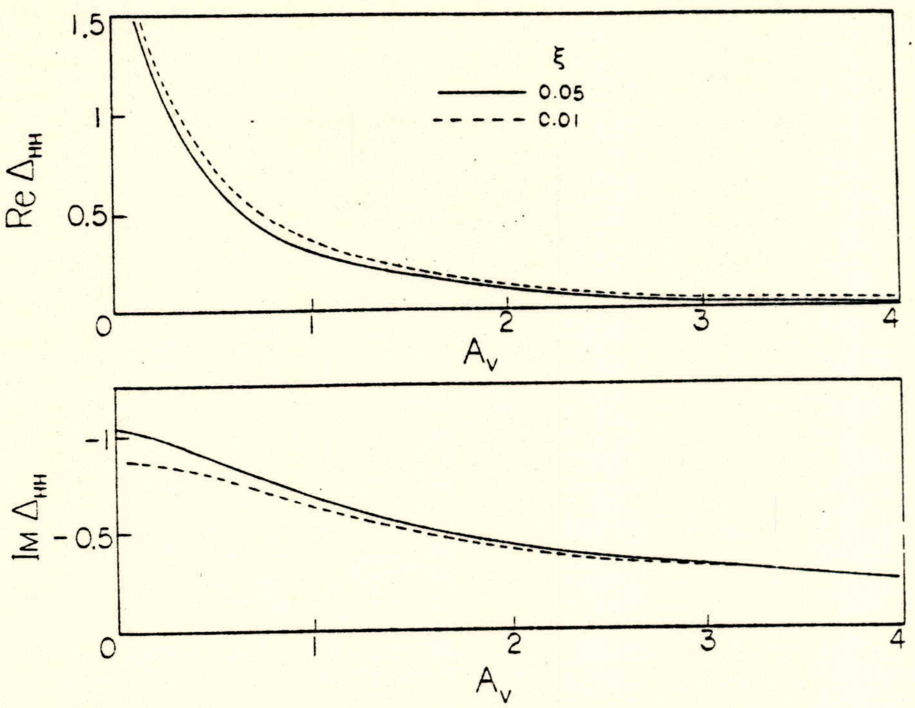

FIG. 6. Effect of soil damping on horizontal compliance (incompressible soil, $n = 1/3$).

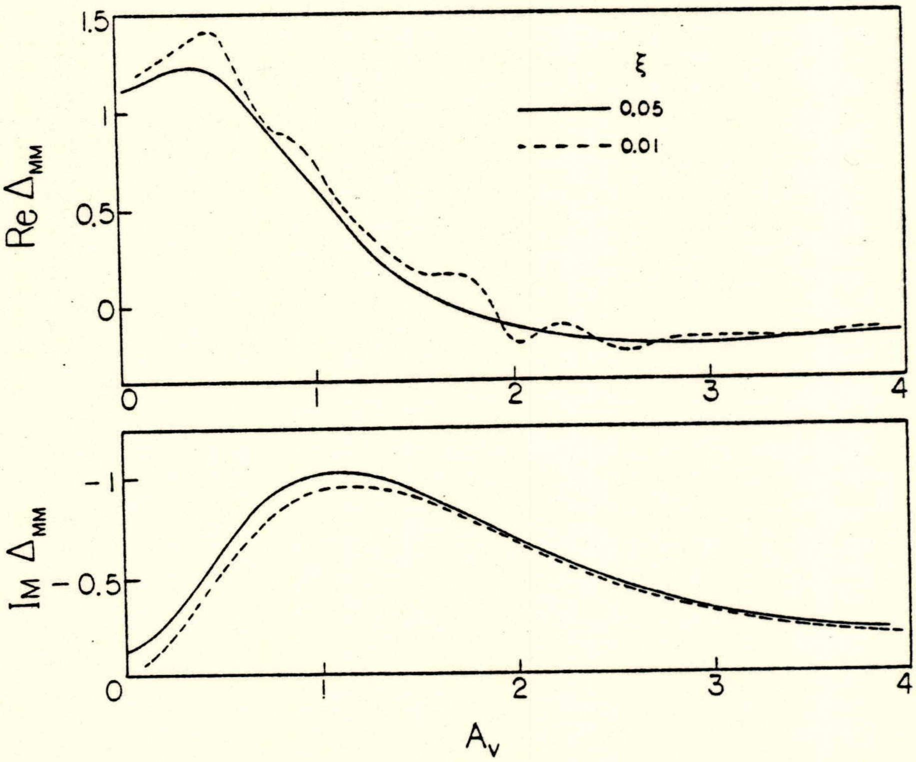

FIG. 7. Effect of soil damping on rocking compliance (incompressible soil, $n = 1/3$).

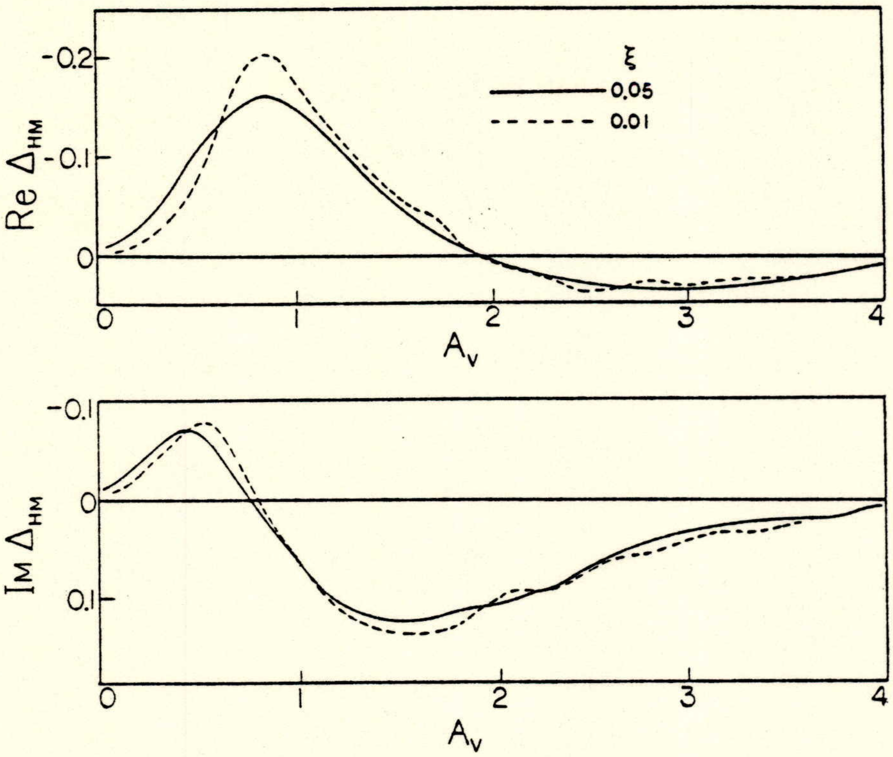

FIG. 8. Effect of soil damping on coupling compliance (incompressible soil, $n = 1/3$).

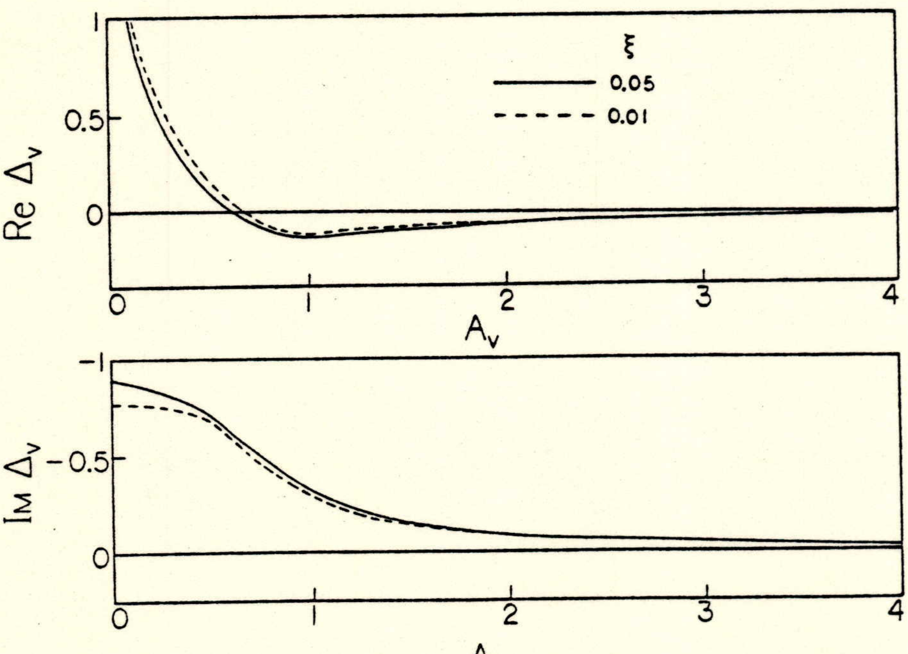

FIG. 9. Effect of soil damping on vertical compliance (incompressible soil, $n = 1/3$).

known for static loading from Gibson [14, 30], but also anticipated in view of the fact that as $n \to 4$ $G_{VH}/E_V \to \infty$, i.e. the material becomes *irrotational* in addition to being *incompressible*.

(b) Both Δ_{HH} and Δ_V display similar asymptotic behaviour for each particular value of n, with their real parts tending to infinity as A_V tends to zero. This is in agreement with classical theory of elasticity according to which one can only specify the surface settlement of a strip loaded halfspace to within an arbitrary displacement [31]. Notice, however, the fast decrease of $\text{Re}\Delta_{HH}$ and $\text{Re}\Delta_V$ with A_V. The corresponding imaginary parts start at a finite "static" value and exhibit a much slower decay with frequency.

The rocking compliances behave differently: their real parts start at a finite static value while the imaginary start from almost zero. This suggests that stresses in a half-space due to moment surface loading are confined to near surface soil with little or no radiation of energy away from the load.

(c) The isotropic halfspace is indeed recovered for $n = 1$; the corresponding compliance functions are in excellent agreement with those reported by Luco and Westman [4].

The influence of the hysteretic soil damping, as demonstrated with Figs. 6–9, is of rather secondary importance, relatively speaking. Reducing the damping factor from $\xi = 5\%$ to $\xi = 1\%$ (a value admittedly very low for most practical geotechnical applications) has mainly two effects: It slightly increases the real components of the compliances while at the same time it decreases their imaginary parts. These effects are most pronounced in case of rocking, especially at the low frequency range.

CONCLUSIONS

The dynamic force-displacement relationships for harmonic motion of a rigid strip footing perfectly bonded to the surface of a viscoelastic cross-anisotropic halfspace have been obtained. The presented solution is analytical in the sense that it is based on a closed-form solution of the governing Navier-type equations for an anisotropic medium. It is also partly numerical in that it determines dynamic flexibility influence coefficients for a set of uniformly spaced nodal points at the surface, by employing a discrete fast Fourier transform technique. An interesting feature of the procedure is that it can be extended to the more general case of a halfspace consisting of any number of cross-anisotropic layers with little additional effort (an attempt already underway on which the Author hopes to report shortly).

Results are presented in the form of dimensionless compliances as functions of a dimensionless frequency factor for the interesting case of an incompressible half-space, which simulates the behaviour of deep clay deposits under "undrained" loading conditions. It is found that, throughout the frequency range examined, the degree of anisotropy may have a profound effect on the dynamic response of foundations. Thus, for a value of the horizontal-to-vertical Young's modulus ratio, $E_H/E_V = 2$, which is typical, e.g. for the heavily overconsolidated London clay, the resulting foundation displacements may be about 30–40% lower than for an isotropic medium ($E_H/E_V = 1$). At high frequency factors, on the other hand, this relation between isotropic and anisotropic displacements is sometimes reversed; thus, in practical situations, careful assessment of the exact effect of anisotropy should be made for the particular frequency range of interest. The effect of soil damping is shown to be of, relatively, secondary importance; in practice, therefore, it is sufficient to obtain a rough estimate of its magnitude on the basis of available empirical data. The reader is cautioned, at the same time, that this may not be true in case of a shallow soil deposit underlain by a rigid, rock-like material; judging from relevant studies with isotropic soils [5, 6], it is anticipated that high damping in such cases would help ameliorate the effect of resonance phenomena that usually take place.

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